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LIPSCHITZ MAPPINGS IN METRIC-LIKE SPACES

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Abstract. Pajoohesh introduced the concept of k-metric spaces and Hiltzler and Seda defined the concept of metric-like spaces. Recently, Kopperman and Pajoohesh proved a fixed point theorem in complete k-metric spaces for a Lipschitz map with bound. In this paper, we prove a fixed point theorem in complete metric-like spaces for a Lipschitz map with bound.

1. Introduction and Preliminaries

Metric spaces has been generalized in many ways. Matthews [8] introduced partial metrics. His goal was to study the reality of finding closer and closer approximation to a given number and showing that contractive algorithms would serve to find these approximations. Moreover, Hitzler and Seda [5] defined the concept of metric-like (or dislocated metric) spaces which generalizes the concept of partial metric spaces. Later, Amini-Harandi [3] established some fixed point theorem in a class of metric-like spaces and many fixed point theorems on metric-like spaces have been proved([1], [2], [4], [6]).

Definition 1.1. Let X be a non-empty set. Then a mapping $d : X\times X \to [0,\infty)$ is called *a metric-like* if for any $x, y, z \in X$, the following conditions hold:
(1) $d(x, y) = 0$ implies $x = y$,
(2) $d(x, y) = d(y, x)$, and
(3) $d(x, z) \leq d(x, y) + d(y, z)$.
In this case, (X, d) is called *a metric-like space.*

In [9], k-metric spaces were defined for some l-group applications, by weaking the metric triangle inequality.

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Definition 1.2. Let X be a non-empty set and k a positive integer. Then a mapping $d : X \times X \to [0, \infty)$ is called *a k-metric* if for any $x, y, z \in X$, the following conditions hold:
(1) $d(x, y) = 0$ if and only if $x = y$,
(2) $d(x, y) = d(y, x)$, and
(3) $d(x, z) \leq k[d(x, y) + d(y, z)]$.
In this case, (X, d) is called *a k-metric space.*

Definition 1.3. [7] Let (X, d) be a k-metric space, q a positive real number and $f : X \to X$ a mapping. Then f is called *a Lipschitz map with bound q* if

$$d(f(x), f(y)) \leq qd(x, y)$$

for all $x, y \in X$.

Kopperman and Pajoohesh [7] proved the fixed point theorem for Lipschitz mapping in complete k-metric spaces.

Now, we define Lipschitz mappings with bound in metric-like spaces.

Definition 1.4. Let (X, d) be a metric-like space, q a positive real number and $f : X \to X$ a mapping. Then f is called *a Lipschitz map with bound q* if

$$d(f(x), f(y)) \leq q|d(x, y) - d(x, x)|$$

for all $x, y \in X$.

In this paper, we will show a fixed point theorem for Lipschitz mappings with bound in complete metric-like spaces and give some examples which are metric-like spaces but not k-metric spaces.

2. Fixed point theorems for Lipschitz mappings with bound

Let (X, d) be a metric-like space. For any $x \in X$ and $\epsilon > 0$, let

$$B_d(x, \epsilon) = \{y \mid |d(x, y) - d(x, x)| < \epsilon\}.$$

Lemma 2.1. [5] *Let (X, d) be a metric-like space. Then we have the following:*
(1) *$\{B_d(x, \epsilon)|x \in X, \epsilon > 0\}$ is a base for some topology τ_d,*
(2) *(X, τ_d) is a T_0-space, and*
(3) *a sequence $\{x_n\}$ converges to x in (X, τ_d) if and only if $\lim_{n\to\infty} d(x_n, x) = d(x, x)$.*

Let (X,d) be a metric-like space. Then
(1) a sequence $\{x_n\}$ is called *a Cauchy sequence* in (X,d) if $\lim_{m,n\to\infty} d(x_n,x_m)$ exists and finite and
(2) (X,d) is called *complete* if every Cauchy sequence in (X,d) is convergent in (X,τ_d).

LEMMA 2.2. *Let (X,d) be a metric-like space. Then we have*

$$|d(x,y)-d(x,x)| \leq d(x,y)$$

for all $x,y\in X$.

Proof. Since $d(x,x)\geq 0$, we have

$$d(x,y)-d(x,x)\leq d(x,y)$$

for all $x,y\in X$ and

$$d(x,y)-d(x,x)\geq d(x,y)-(d(x,y)+d(x,y))=-d(x,y)$$

for all $x,y\in X$. Hence one has the result. □

By Lemma 2.2 and induction, we have the following lemma.

LEMMA 2.3. *Let (X,d) be a metric-like space and $f:X\to X$ a Lipschitz map with bound q. Then for any $x\in X$ and any $n\in\mathbb{N}$, the following inequality holds:*

$$d(x,f^n(x))\leq \sum_{i=0}^{n-1} q^i d(x,f(x)). \tag{2.1}$$

Proof. Clearly, (2.1) holds for $n=1$. Suppose that (2.1) holds for some $n\in\mathbb{N}$ with $n\geq 2$. Then by Lemma 2.2, we have

$$\begin{aligned} d(x,f^{n+1}(x)) &\leq d(x,f(x))+d(f(x),f^{n+1}(x)) \\ &\leq d(x,f(x))+q\Big|d(x,f^n(x))-d(x,x)\Big| \\ &\leq d(x,f(x))+qd(x,f^n(x)) \\ &\leq d(x,f(x))+q\sum_{i=1}^{n} q^i d(x,f(x)) \\ &= \sum_{i=0}^{n} q^i d(x,f(x)). \end{aligned}$$

By induction, we have the result. □

LEMMA 2.4. *Let (X, d) be a metric-like space and $f : X \to X$ a Lipschitz map with bound q. Then for any $\epsilon > 0$, there is a $\delta > 0$ such that*

$$f[B_d(x, \delta)] \subseteq B_d(f(x), \epsilon)].$$

Proof. Let $\epsilon > 0$ and $\delta = \frac{\epsilon}{q}$. Then $\delta > 0$ and take any $y \in B_d(x, \delta)$. Since $|d(x, y) - d(x, x)| < \delta$, by Lemma 2.2,

$$\begin{aligned} |d(f(x), f(y)) - d(f(x), f(x))| &\leq d(f(x), f(y)) \\ &\leq q\Big|d(x, y) - d(x, x)\Big| \\ &< q\delta = \epsilon \end{aligned}$$

and hence we have the result. □

Using Lemma 2.2, Lemma 2.3, and Lemma 2.4, we have the following fixed theorem.

THEOREM 2.5. *Let (X, d) be a metric-like space and $f : X \to X$ a Lipschitz map with bound $q < 1$. Then f has the unique fixed point z in X with $d(z, z) = 0$.*

Proof. Let $x \in X$ and $\epsilon > 0$. Since $0 < q < 1$, there is a positive integer l such that

$$\frac{q^l}{1-q} d(x, f(x)) < \epsilon. \tag{2.2}$$

For $m > n \geq l$, by Lemma 2.2, (2.2), and Lemma 2.3, we have

$$\begin{aligned} d(f^n(x), f^m(x)) &\leq q\Big|d(f^{n-1}(x), f^{m-1}(x)) - d(f^{n-1}(x), f^{n-1}(x))\Big| \\ &\leq q d(f^{n-1}(x), f^{m-1}(x)) \\ &\leq q^n d(x, f^{m-n}(x)) \\ &\leq q^n \sum_{i=0}^{m-n-1} q^i d(x, f(x)) \\ &< \frac{q^n}{1-q} d(x, f(x)) < \epsilon. \end{aligned}$$

Hence $\{f^n(x)\}$ is a Cauchy sequence in (X, d) and since (X, d) is a complete metric-like space, there is an $y \in X$ such that

$$\lim_{n\to\infty} d(f^n(x), y) = d(y, y).$$

Now, we claim that $\lim_{n\to\infty} d(f^n(x), f(y)) = d(f(y), f(y))$. Let $\delta > 0$. By Lemma 2.4, there is an $\eta > 0$

$$f[B_d(y,\eta)] \subseteq B_d(f(y),\delta)]. \tag{2.3}$$

Since $\lim_{n\to\infty} d(f^n(x), y) = d(y, y)$, there is a natural number l_0 such that for any $n \geq l_0$,

$$|d(f^n(x), y) - d(y, y)| < \eta$$

and by (2.3), for any $n \geq l_0$, we get

$$|d(f^{n+1}(x), f(y)) - d(f(y), f(y))| < \delta.$$

Hence we have

$$\lim_{n\to\infty} d(f^n(x), f(y)) = d(f(y), f(y)). \tag{2.4}$$

Let $z = f(y)$. Since $f : X \to X$ is a Lipschitz map with bound q,

$$d(z, z) = d(f(y), f(y)) \leq q|d(y, y) - d(y, y)| = 0$$

and hence $d(z, z) = 0$. Thus by (2.4), we get

$$\lim_{n\to\infty} d(f^n(x), z) = 0 \tag{2.5}$$

Since $d(z, z) = 0$, we get

$$\begin{aligned} d(f(z), z) &\leq d(f(z), f^{n+1}(x)) + d(z, f^{n+1}(x)) \\ &\leq q|d(z, f^n(x)) - d(z, z)| + d(z, f^{n+1}(x)) \\ &\leq qd(z, f^n(x)) + d(z, f^{n+1}(x)). \end{aligned} \tag{2.6}$$

Letting $n \to \infty$ in (2.6), by (2.5), we have

$$d(f(z), z) = 0$$

and so $f(z) = z$. Thus z is a fixed point of f with $d(z, z) = 0$.

To show the uniquness of z, let w be another fixed point of f with $d(w, w) = 0$. Then

$$d(z, w) = d(f(z), f(w)) \leq q|d(z, w) - d(z, z)| = qd(z, w)$$

and since $0 < q < 1$, $d(z, w) = 0$. Henec $z = w$. □

We will give an example which is complete metric-like but not k-metric.

Example 2.6. *Let $X = \{0, 1, 2\}$ and $d : X \times X \to [0, \infty)$ be a map defined by*

$$d(0,0) = d(1,1) = 0,\ d(0,1) = d(1,0) = 1,$$
$$d(0,2) = d(2,0) = \frac{3}{2},\ d(1,2) = d(2,1) = \frac{8}{5},\ d(2,2) = \frac{1}{2}.$$

Then (X, d) is a metric-like space but it is not a metric space. Hence (X,d) is not an 1-metric space. Clearly, (X, d) is a complete metric-like space.

Suppose that f is a Lipschitz mapping with bound $q < 1$. Then by Theorem 2.5, there is the unique fixed point z of f with $d(z, z) = 0$. By the definition of d, $z = 0$ or $z = 1$. If $z = 1$, then

$$d(f(0), 1) = d(f(0), f(1)) \leq q|d(0, 1) - d(0, 0)| = q < 1,$$

$$d(f(0), 0) \leq d(f(0), f(1)) + d(1, 0) = d(f(0), 1) = q < 1,$$

and hence $d(f(0), 1) = d(f(0), 0) = 0$. That is, $1 = f(0) = 0$ which is a contradiction and thus $z = 0$. Since

$$d(f(1), 0) = d(f(1), f(0)) \leq q|d(1, 0) - d(0, 0)| = q < 1,$$

and so $f(1) = 0$.

Thus $f : X \to X$ is a Lipschitz mapping with bound $q < 1$ if and only if $f(0) = f(1) = 0$.

Example 2.7. *Let $A = \{2n|n \in \mathbb{N}\}$, $B = \mathbb{N} - A$ and $d : \mathbb{N} \times \mathbb{N} \to [0, \infty)$ a map defined by*

$$d(x, y) = \begin{cases} 1 - \frac{2}{x}, & \text{if } x = y \in A, \\ 0, & \text{if } x = y \in B, \\ 1 + \frac{1}{x} + \frac{1}{y}, & \text{if } x \neq y \end{cases}$$

Then $(\mathbb{N}, d)$ is a metric-like space but it is not a metric space. Hence $(\mathbb{N}, d)$ is not an 1-metric space.

Now, we claim that $(\mathbb{N}, d)$ is a complete metric-like space. Let $\{x_n\}$ be a Cauchy sequence in $(\mathbb{N}, d)$. Then there is an $l \in \mathbb{N}$ such that for any $n, m \geq l$,

$$d(x_n, x_m) < \frac{1}{4}$$

and by the definition of d, for any $n, m \geq l$,

$$x_n = x_m \in B \ \text{ or } \ x_n = x_m = 2.$$

Suppose that there are $n, m \geq l$ *such that* $x_m \in B$ *and* $x_n = 2$. *Then* $d(x_n, x_m) > 1$ *which is a contradiction. Hence either for any* $n \geq l$, $x_n = 2t - 1$ *for some* $t \in \mathbb{N}$ *or for any* $n \geq l$, $x_n = 2$. *Thus either*

$$\lim_{n\to\infty} d(x_n, 2k-1) = d(2k-1, 2k-1) = 0$$

or

$$\lim_{n\to\infty} d(x_n, 2) = d(2, 2) = 0.$$

Hence $\{x_n\}$ *is convergent in* $(\mathbb{N}, d)$ *and thus* $(\mathbb{N}, d)$ *is a complete metric-like space.*

Let $f : \mathbb{N} \to \mathbb{N}$ *be a Lipschitz mapping with bound* q $(0 < q < 1)$. *By Theorem 2.5, there is a fixed point* z *of* f *with* $d(z, z) = 0$. *Since* $0 < q < 1$, *there is an* l *such that* $3q^l < \frac{1}{4}$. *For* $n \in \mathbb{N}$, *by Lemma 2.2, we have*

$$d(f^l(n), z) = d(f^l(n), f^l(z)) \leq q^l d(n, z) < 3q^l < \frac{1}{4},$$

because $d(a, b) < 3$ *for all* $a, b \in \mathbb{N}$. *Hence* $f^l(n) = z$ *and thus* f^l *is a constant map.*

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